Journal of Advanced Physics
Vol. 6, pp. 430–433, 2017
(www.aspbs.com/jap)

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Remarks on Fractional Hamilton-Jacobi Formalism with Second-Order Discrete Lagrangian Systems

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In this paper, we examined the Hamilton-Jacobi formulation for discrete Lagrangian systems containing second-order fractional derivatives. The fractional Euler-Lagrange equations for the calculus of variations problem for these systems are analyzed. The Hamilton's equations of motion for these systems are derived. The equivalence between the fractional Lagrangian and the fractional Hamiltonian formalism are achieved. We have examined one example to illustrate the formalism.

KEYWORDS: Fractional Calculus, Fractional Variational Problem, Euler-Lagrange Equation, Second-Order Lagrangian, Fractional Hamilton-Jacobi Formalism.

1. INTRODUCTION

The study of fractional calculus has become a novel area in various branches of science, applied mathematics, physical systems and engineering,[1–8] starting from the early development by Riewe[9,10] of the generalized Hamiltonian formulation, then this formalism has found a wide range of applications.[1–8] Riewe has studied non-conservative Lagrangian and Hamiltonian mechanics within fractional calculus.[9,10] He has used the fractional calculus to obtain a formalism which can be applied for both conservative and non-conservative systems. One can obtain the Lagrangian and the Hamiltonian equations of motion for the non-conservative systems. Besides, the generalization of Lagrangian and Hamiltonian fractional mechanics with fractional derivatives were extended and discussed in details.[11–17]

The formalism for investigating the fractional variational problem of Lagrange represents an important part of fractional calculus and it was discussed by Agrawal,[11,12] and this formalism can be extended to Lagrangians systems with higher derivatives. Theories associated with higher-order Lagrangians systems have been discussed by Ostrogradski. In the Ostrogradski's formalism, the Euler's and Hamilton's equations of motion were derived and then, his formalism was extended to singular systems.[18–23] Recently, the fractional Hamiltonian analysis for higher-order derivatives systems were investigated for nonsingular systems and the generalization of Ostrogradski's formulation was discussed within framework of fractional calculus.[24]

This paper is organized as follows: In Section 2, the basic definitions of fractional derivatives are reviewed briefly. In Section 3, the fractional variational problem with second-order Lagrangians are investigated and the Euler-Lagrange equations are derived. In Section 4, the fractional Hamiltonian formalism for second-order Lagrangian system is investigated. In Section 5, one illustrative example is examined. The work closes with some concluding remarks in Section 6.

2. BASIC DEFINITIONS

In this section, we briefly review some fundamental definitions of the fractional derivative in Agrawal works.[11,12] The left Riemann-Liouville fractional derivative is defined as

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\,d\tau \quad (1)$$

and the right Riemann-Liouville fractional derivative has the form

$$ {}_tD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^n \int_t^b (\tau-t)^{n-\alpha-1} f(\tau)\,d\tau \quad (2)$$

Where α represents the order of derivative such that $n-1 \leq \alpha \leq n$. If α is an integer, these derivatives are defined as follows

$$ \begin{aligned} {}_aD_t^\alpha f(t) &= \left(\frac{d}{dt}\right)^\alpha f(t) \\ {}_tD_b^\alpha f(t) &= \left(-\frac{d}{dt}\right)^\alpha f(t), \quad \alpha = 1, 2, \ldots \end{aligned} \quad (3)$$

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Received: 5 October 2016
Accepted: 18 October 2016

The RL fractional derivatives have the general properties can be written as

$$ {}_aD_t^p({}_aD_t^{-q}f(t)) = {}_aD_t^{p-q}f(t) \tag{4a}$$

under the assumptions that $f(t)$ is continuous and $p \geq q \geq 0$. For $p > 0$ and $t > a$, we get

$$ {}_aD_t^p({}_aD_t^{-p}f(t)) = f(t) \tag{4b}$$

the general formula of semi-group property is written as[2]

$$ {}_aD_t^\alpha({}_aD_t^\beta)f(t) = {}_aD_t^{\alpha+\beta}f(t) \tag{4c}$$

Let f and g are two continuous functions on $[a, b]$. Then, for all $t \in [a, b]$, the following property holds: For

$$ m>0, \quad \int_a^b ({}_aD_t^m f(t))g(t)\,dt = \int_a^b f(t)({}_xD_b^m g(t))\,dt \tag{4d}$$

3. THE FRACTIONAL VARIATIONAL PROBLEM WITH SECOND-ORDER LAGRANGIANS

The Lagrangian of the fractional calculus of variational problem has the form

$$ L(q, {}_aD_t^\alpha q, {}_tD_b^\beta q, {}_aD_t^{2\alpha} q, {}_tD_b^{2\beta} q, t) \tag{5}$$

All functions $q(t)$ have continuous LRLFD of order α and RRLFD of order β for $a \leq t \leq b$, and satisfy the boundary conditions

$$ \begin{gathered} q(a) = q_a, \quad q(b) = q_b, \quad {}_aD_t^\alpha q(a) = {}_aD_t^\alpha q_a, \\ {}_tD_b^\beta q(b) = {}_tD_b^\beta q_b \end{gathered} \tag{6}$$

Let us examine the extrema of the functional

$$ \begin{aligned} &S[q, {}_aD_t^\alpha q, {}_tD_b^\beta q] \\ &= \int L(q, {}_aD_t^\alpha q, {}_tD_b^\beta q, {}_aD_t^{2\alpha} q, {}_tD_b^{2\beta} q, t)\,dt \end{aligned} \tag{7}$$

Where $0 < \alpha, \beta \leq 1$ and $\alpha, \beta \in R^+$, the simplest variational problem can be obtained when α and β of our problem are equal unity.

Let us define a family of functions for the necessary conditions for the extremum of the action (7) in the form read as

$$ q(t) = q^*(t) + \epsilon\eta(t) \tag{8}$$

where $q^*(t)$ is satisfying the extremum of the action (7) and it is defined as real function. $\epsilon \in R$ is a constant, and the function η defined in $[a, b]$ satisfy the boundary conditions

$$ \eta(a) = \eta(b) = 0 \tag{9a}$$

$$ \dot{\eta}(a) = \dot{\eta}(b) = 0 \tag{9b}$$

We shall define the set of linear operators as follows

$$ {}_aD_t^\alpha q(t) = {}_aD_t^\alpha q^*(t) + \epsilon_a D_t^\alpha \eta(t) \tag{10a}$$

$$ {}_tD_b^\beta q(t) = {}_tD_b^\beta q^*(t) + \epsilon_t D_b^\beta \eta(t) \tag{10b}$$

$$ {}_aD_t^{2\alpha} q(t) = {}_aD_t^{2\alpha} q^*(t) + \epsilon_a D_t^{2\alpha} \eta(t) \tag{10c}$$

$$ {}_tD_b^{2\beta} q(t) = {}_tD_b^{2\beta} q^*(t) + \epsilon_t D_b^{2\beta} \eta(t) \tag{10d}$$

Substituting Eqs. (8) and (10) into Eq. (7), one can find for each $\eta(t)$

$$ \begin{aligned} S[\in] = \int_a^b L(t, &q* + \in\eta, {}_aD_t^\alpha q* + \in {}_aD_t^\alpha \eta, \\ &{}_tD_b^\beta q* + \in {}_tD_b^\beta \eta, {}_aD_t^{2\alpha} q* + \in {}_aD_t^{2\alpha}\eta \\ &+ {}_tD_b^{2\beta} q* + \in {}_tD_b^{2\beta}\eta)\,dt \end{aligned} \tag{11}$$

is a function of $\in$ only. One can note that the action $S(\in)$ is extremum at $\in = 0$.

Now, differentiate Eq. (11) with respect to $\in$; we can obtain the variation of the action $S[q, {}_aD_t^\alpha q, {}_tD_b^\beta q]$ at $q(t)$ along $\eta(t)$

$$ \begin{aligned} \frac{dS}{d\in} = \int_a^b \Bigg[&\frac{\partial L}{\partial q}\eta + \frac{\partial L}{\partial_a D_t^\alpha q} {}_aD_t^\alpha \eta + \frac{\partial L}{\partial_t D_b^\beta} {}_tD_b^\beta \eta \\ &+ \frac{\partial L}{\partial_a D_t^{2\alpha} q} {}_aD_t^{2\alpha} \eta + \frac{\partial L}{\partial_t D_b^{2\beta} q} {}_tD_b^{2\beta}\eta \Bigg] dt \end{aligned} \tag{12}$$

Now, we shall examine the extremum condition for the action $S[\in]$ to have an extremum is that $dS/d\in$ must be zero.

$$ \begin{aligned} \int_a^b \Bigg[&\frac{\partial L}{\partial q}\eta + \frac{\partial L}{\partial_a D_t^\alpha q} {}_aD_t^\alpha \eta + \frac{\partial L}{\partial_t D_b^\beta} {}_tD_b^\beta \eta \\ &+ \frac{\partial L}{\partial_a D_t^{2\alpha} q} {}_aD_t^{2\alpha} \eta + \frac{\partial L}{\partial_t D_b^{2\beta} q} {}_tD_b^{2\beta}\eta \Bigg] dt = 0 \end{aligned} \tag{13}$$

Using the formula for fractional integration by parts for the second integral in Eq. (13), one can write[2]

$$ \int_a^b \frac{\partial L}{\partial_a D_t^\alpha q} {}_aD_t^\alpha \eta\,dt = \int_a^b {}_tD_b^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha q} \right) \eta\,dt \tag{14a}$$

provided that $\partial L/\partial_a D_t^\alpha q$ or η is zero at $t = a$ and $t = b$. By using Eq. (9a), this condition is satisfied, and it follow that Eq. (14a) is valid. Similarly, the third, fourth and fifth integrals in Eq. (13) can be written as

$$ \int_a^b \frac{\partial L}{\partial_t D_b^\beta q} {}_tD_b^\beta \eta\,dt = \int_a^b {}_aD_t^\beta \left(\frac{\partial L}{\partial_t D_b^\beta q} \right) \eta\,dt \tag{14b}$$

$$ \int_a^b \frac{\partial L}{\partial_a D_t^{2\alpha} q} {}_aD_t^{2\alpha} \eta\,dt = \int_a^b {}_tD_b^{2\alpha} \left(\frac{\partial L}{\partial_a D_t^{2\alpha} q} \right) \eta\,dt \tag{14c}$$

$$ \int_a^b \frac{\partial L}{\partial_t D_b^{2\beta} q} {}_tD_b^{2\beta} \eta dt = \int_a^b {}_aD_t^{2\beta} \left(\frac{\partial L}{\partial_t D_b^{2\beta} q} \right) \eta\,dt \tag{14d}$$

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Substituting Eq. (14) into Eq. (13), we get

$$\int_a^b \left[\frac{\partial L}{\partial q} + {}_tD_b^\alpha \frac{\partial L}{\partial_a D_t^\alpha q} + {}_aD_t^\beta \frac{\partial L}{\partial_t D_b^\beta} + {}_tD_b^{2\alpha} \frac{\partial L}{\partial_a D_t^{2\alpha} q} {}_a + {}_aD_t^{2\beta} \frac{\partial L}{\partial_t D_b^{2\beta} q}\right]\eta\, dt = 0 \qquad (15)$$

Since η is arbitrary, it follows that

$$\frac{\partial L}{\partial q} + {}_tD_b^\alpha \frac{\partial L}{\partial_a D_t^\alpha q} + {}_aD_t^\beta \frac{\partial L}{\partial_t D_b^\beta q} + {}_tD_b^{2\alpha} \frac{\partial L}{\partial_a D_t^{2\alpha} q} + {}_aD_t^{2\beta} \frac{\partial L}{\partial_t D_b^{2\beta} q} = 0 \qquad (16)$$

Equation (16) is the formulation of Euler-Lagrange equation for the fractional calculus of variations problem with second-order derivatives.

4. FRACTIONAL HAMILTONIAN FORMALISM

The fractional Lagrangian for discrete systems can be defined as follow:

$$L(q, {}_aD_t^\alpha q, {}_tD_b^\beta q, {}_aD_t^{2\alpha} q, {}_tD_b^{2\beta} q, t) \qquad (17)$$

The corresponding Euler-Lagrange equation for the fractional calculus of variational problem are obtained from extremization the action

$$S[q, {}_aD_t^\alpha q, {}_tD_b^\beta q] = \int L(q, {}_aD_t^\alpha q, {}_tD_b^\beta q, {}_aD_t^{2\alpha} q, {}_tD_b^{2\beta} q, t)\, dt \qquad (18)$$

and has the form

$$\frac{\partial L}{\partial q} + {}_tD_b^\alpha \frac{\partial L}{\partial_a D_t^\alpha q} + {}_aD_t^\beta \frac{\partial L}{\partial_t D_b^\beta q} + {}_tD_b^{2\alpha} \frac{\partial L}{\partial_a D_t^{2\alpha} q} + {}_aD_t^{2\beta} \frac{\partial L}{\partial_t D_b^{2\beta} q} = 0 \qquad (19)$$

The fractional canonical momenta are written as[24]

$$p_\alpha = \frac{\partial L}{\partial_a D_t^\alpha q} - {}_aD_t^\alpha\left(\frac{\partial L}{\partial_a D_t^{2\alpha} q}\right), \qquad (20a)$$

$$p_\beta = \frac{\partial L}{\partial_t D_b^\beta q} - {}_tD_b^\beta\left(\frac{\partial L}{\partial_t D_b^{2\beta} q}\right), \qquad (20b)$$

$$\pi_\alpha = \frac{\partial L}{\partial_a D_t^{2\alpha} q}, \qquad (20c)$$

$$\pi_\beta = \frac{\partial L}{\partial_t D_b^{2\beta} q} \qquad (20d)$$

The fractional Hamiltonian can be written as

$$H = p_\alpha {}_aD_t^\alpha q + p_\beta {}_tD_b^\beta q + \pi_\alpha {}_aD_t^{2\alpha} q + \pi_\beta {}_tD_b^{2\beta} q - L \qquad (21)$$

The total differential of this Hamiltonian is given by

$$dH = p_\alpha d_a D_t^\alpha q + dp_\alpha {}_aD_t^\alpha q + p_\beta d_t D_b^\beta q + dp_\beta {}_tD_b^\beta q + \pi_\alpha d_a D_t^{2\alpha} q + {}_aD_t^{2\alpha} q\, d\pi_\alpha + \pi_\beta d_t D_b^{2\beta} q + {}_tD_b^{2\beta} q\, d\pi_\beta - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial_a D_t^\alpha q} d_a D_t^\beta q - \frac{\partial L}{\partial_t D_b^\beta q} d_t D_b^\beta q - \frac{\partial L}{\partial_a D_t^{2\alpha} q} d_a D_t^{2\alpha} q - \frac{\partial L}{\partial_t D_b^{2\beta} q} d_t D_b^{2\beta} q - \frac{\partial L}{\partial t} dt \qquad (22)$$

Substituting the values of the momenta from Eq. (20) and the value of $\partial L/\partial q$ from Eq. (19) in Eq. (22), we get

$$dH = dp_\alpha {}_aD_t^\alpha q + dp_\beta {}_tD_b^\beta q + {}_aD_t^{2\alpha} q\, d\pi_\alpha + {}_tD_b^{2\beta} q\, d\pi_\beta - {}_aD_t^\alpha \pi_\alpha d_a D_t^\alpha q - {}_tD_b^\beta \pi_\beta d_t D_b^\beta q + [{}_tD_b^\beta(p_\alpha + {}_aD_t^\alpha \pi_\alpha) + {}_aD_t^\alpha(p_\beta + {}_tD_b^\beta \pi_\beta) + {}_tD_b^{2\alpha}\pi_\alpha + {}_aD_t^{2\beta}\pi_\beta]\, dq - \frac{\partial L}{\partial t} dt \qquad (23)$$

The Hamiltonian function is defined as

$$H = (q, {}_aD_t^\alpha q, {}_tD_b^\beta q, p_\alpha, p_\beta, \pi_\alpha, \pi_\beta, t) \qquad (24)$$

Thus, the total differential of this function gives

$$dH = \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial_a D_t^\alpha q} d_a D_t^\alpha q + \frac{\partial H}{\partial_t D_b^\beta q} d_t D_b^\beta q + \frac{\partial H}{\partial p_\alpha} dp_\alpha + \frac{\partial H}{\partial p_\beta} dp_\beta \frac{\partial H}{\partial \pi_\alpha} d\pi_\alpha + \frac{\partial H}{\partial \pi_\beta} d\pi_\beta + \frac{\partial H}{\partial t} dt \qquad (25)$$

Comparing Eqs. (23) and (25), we get the following Hamilton's equations of motion and they can be written as

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \quad \frac{\partial H}{\partial p_\alpha} = {}_aD_t^\alpha q, \quad \frac{\partial H}{\partial p_\beta} = {}_tD_b^\beta q,$$
$$\frac{\partial H}{\partial_a D_t^\alpha q} = -{}_aD_t^\alpha \pi_\alpha, \quad \frac{\partial H}{\partial_t D_b^\beta q} = -{}_tD_b^\beta \pi_\beta,$$
$$\frac{\partial H}{\partial \pi_\alpha} = {}_aD_t^{2\alpha} q, \quad \frac{\partial H}{\partial \pi_\beta} = {}_tD_b^{2\beta} q$$
$$\frac{\partial H}{\partial q} = [{}_tD_b^\beta(p_\alpha + {}_aD_t^\alpha \pi_\alpha) + {}_aD_t^\alpha(p_\beta + {}_tD_b^\beta \pi_\beta) + {}_tD_b^{2\alpha}\pi_\alpha + {}_aD_t^{2\beta}\pi_\beta] \qquad (26)$$

It is interesting to notice that there are equivalence between the fractional Lagrangian Eq. (16) and the fractional Hamiltonian formalism Eq. (26). In other words, one can obtain the same results for the equations of motion using the two formalisms.

5. EXAMPLE

As a first example let us start with the following second-order regular (discrete) Lagrangian:[21, 25]

$$L = \frac{1}{2}(\ddot{q}^2 - \dot{q}^2) \tag{27}$$

The corresponding fractional Lagrangian

$$L' = \frac{1}{2}({}_aD_t^{2\alpha}q)^2 - \frac{1}{2}({}_aD_t^{\alpha}q)^2 \tag{28}$$

The corresponding Euler-Lagrange equation corresponding to Eq. (28) becomes

$$-{}_tD_b^{\alpha}\,{}_aD_t^{\alpha}q + {}_tD_b^{2\alpha}\,{}_aD_t^{2\alpha}q = 0 \tag{29}$$

For $\alpha \to 1$, this equation leads to the classical solution

$$q = A + Bt + C\cos t + D\sin t \tag{30}$$

The fractional canonical momenta are written as

$$\begin{aligned} p_\alpha &= -{}_aD_t^{\alpha}q - {}_aD_t^{3\alpha}q, \quad p_\beta = 0, \\ \pi_\alpha &= {}_aD_t^{2\alpha}q, \quad \pi_\beta = 0 \end{aligned} \tag{31}$$

The fractional Hamiltonian reads as

$$H = p_\alpha\,{}_aD_t^{\alpha}q + \frac{1}{2}(\pi_\alpha)^2 + \frac{1}{2}({}_aD_t^{\alpha}q)^2 \tag{32}$$

One can obtain the same result of Eq. (29) using Hamilton's equations of motion Eq. (26)

$$\begin{aligned} &\frac{\partial H}{\partial p_\alpha} = {}_aD_t^{\alpha}q, \quad \frac{\partial H}{\partial p_\beta} = {}_tD_b^{\beta}q = 0, \\ &\frac{\partial H}{\partial_a D_t^{\alpha}q} = -{}_aD_t^{\alpha}\pi_\alpha = p_\alpha + {}_aD_t^{\alpha}q, \\ &\frac{\partial H}{\partial_t D_b^{\beta}q} = -{}_tD_b^{\beta}\pi_\beta = 0, \\ &\frac{\partial H}{\partial \pi_\alpha} = {}_aD_t^{2\alpha}q = \pi_\alpha, \quad \frac{\partial H}{\partial \pi_\beta} = {}_tD_b^{2\beta}q = 0 \\ &\frac{\partial H}{\partial q} = {}_tD_b^{\beta}(p_\alpha + {}_aD_t^{\alpha}\pi_\alpha) + {}_tD_b^{2\alpha}\pi_\alpha = 0 \\ &\quad = -{}_tD_b^{\beta}\,{}_aD_t^{\alpha}q + {}_tD_b^{2\alpha}\,{}_aD_t^{2\alpha}q \end{aligned} \tag{33}$$

It observed that the fractional Lagrangian Eq. (29) is in exact agreement with the fractional Hamiltonian formulation Eq. (33).

Also, we conclude that the classical solutions are obtained if α and β are both equal unity in Eq. (33) and they have the same results of Eq. (30).

6. CONCLUSION

Following to Agrawal works, we have considered the generalized mechanics to obtain the fractional Hamiltonian formulation for second-order fractional discrete Lagrangian systems using the calculus of variations. The fractional Euler-Lagrange equations for these systems were derived. We have constructed the fractional Hamiltonian for these systems. The Hamiltonian equations of motion for these systems were obtained in a similar manner to the usual mechanics. It was proven that there were equivalence between the Lagrangian containing fractional derivatives and the fractional Hamiltonian formalism. One illustrative example was discussed.

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